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INTERSECTION PROPERTIES OF BOXES IN $\mathbf{R}^d$

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Dedicated to Tibor Gallai on his seventieth birthday

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A family of sets is called n-pierceable if there exists a set of n points such that each member of the family contains at least one of the points. Helly's theorem on intersections of convex sets concerns 1-pierceable families. Here the following Helly-type problem is investigated: If d and n are positive integers, what is the least $h=h(d, n)$ such that a family of boxes (with parallel edges) in d-space is n-pierceable if each of its h-membered subfamilies is n-pierceable? The somewhat unexpected solution is: (i) $h(d, 2)$ equals $3d$ for odd d and $3d-1$ for even d; (ii) $h(2, 3)=16$; and (iii) $h(d, n)$ is infinite for all (d, n) with $d \geqq 2$ and $n \geqq 3$ except for $(d, n)=(2, 3)$.

1. Introduction

Let $\mathscr{F}$ be a family of subsets of a space X and let n and k be cardinals, n finite. We shall use the following terminology:

$\mathscr{F} \in \Pi^n$ (read: $\mathscr{F}$ is n-pierceable, or pierceable by n points) iff there exists a set $A \subset X$ consisting of n or fewer points such that $A \cap F \neq \emptyset$ for each F from $\mathscr{F}$.

$\mathscr{F} \in \Pi_k$ iff each subfamily of $\mathscr{F}$ consisting of less than $k+1$ members is pierceable by one point (that is: has a non-empty intersection).

$\mathscr{F} \in \Pi_k^n$ iff each subfamily of $\mathscr{F}$ consisting of less than $k+1$ members is n-pierceable.

Obviously $\Pi_k = \Pi_k^1$ and $\Pi_k^n \supset \Pi_{k+1}^n \supset \ldots \supset \Pi_{\aleph_0}^n \supset \Pi^n$; if $\mathscr{F}$ consists of compact sets then $\Pi_{\aleph_0}^n = \Pi^n$.

Let Φ denote a class of families $\mathscr{F}$. We are interested in the Helly-type problem (see [1] p. 127) of determining $h(\Phi, n)$, where $h(\Phi, n)$ is defined as the smal-lestcardinal h with the property

$$\Pi_h^n \cap \Phi = \Pi^n \cap \Phi.$$

In other words, h is the smallest cardinal such that

$$\mathscr{F} \in \Phi \cap \Pi_h^n \quad \text{implies} \quad \mathscr{F} \in \Pi^n.$$

For example, if Z^d denotes the class of all families of compact convex subsets of the d-dimensional real affine space $\mathbf{R}^d$, Helly's theorem on intersections of convex sets (see [1] for references and for related results) becomes $h(Z^d, 1)=d+1$.

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It is well known (see [4], p. 17, or [5], p. 12) that $h(Z^d, n) = \aleph_0$ if $d \geqq 2$ and $n \geqq 2$.

The aim of the present paper is to establish the values of the function h defined by $h(d, n) := h(\Delta^d, n)$, where Δ^d denotes the class of all families of boxes in $\mathbf{R}^d$ with edges parallel to the coordinate axes. Our result may be formulated in the following manner (d and n denoting positive integers):

Theorem. (i) $h(d, 1) = 2$ *for all* $d (d \geqq 1)$;

(ii) $h(1, n) = n+1$ *for all* n;

(iii) $h(d, 2) = \begin{cases} 3d & \textit{for odd } d; \\ 3d-1 & \textit{for even } d; \end{cases}$

(iv) $h(2, 3) = 16$;

(v) $h(d, n) = \aleph_0$ *for* $d \geqq 2$, $n \geqq 3$, *and* $(d, n) \neq (2, 3)$.

The following table summarizes the values of the function h:

d / n	1	2	3	4	5	6	$2m$	$2m+1$
1	2	2	2	2	2	2	2	2
2	3	5	9	11	15	17	$6m-1$	$6m+3$
3	4	16						
4	5							
5	6				$\aleph_0$			
n	$n+1$							

The proof of the various parts of the theorem is given in the following four sections; Section 6 is devoted to remarks and open problems.

2. Proof of parts (i) and (ii) of the theorem

Since for each d there exist $n+1$ disjoint boxes in $\mathbf{R}^d$ it is obvious that $h(d, n) \geqq n+1$ for all d and n.

On the other hand, projecting the boxes into the coordinate axes and using Helly's theorem for the real line yields the well known result $h(d, 1) \leqq 2$, which — together with the above remark — completes the proof of part (i) of the theorem.

In order to establish (ii) we shall prove by induction on n that $h(1, n) \leqq n+1$. In view of (i) we may assume $n > 1$. Let a non-empty $\mathscr{F}$ be given and $\mathscr{F} \in \Pi^n_{n+1} \cap \Delta^1$; the left endpoints of the (compact) segments in $\mathscr{F}$ form a non-empty set bounded from above, say with least upper bound a. Let $\mathscr{F}^*$ be the family consisting of all those members of $\mathscr{F}$ which do not contain a. Then clearly $\mathscr{F}^* \in \Pi^{n-1}_n \cap \Delta^1$; hence $\mathscr{F}^*$ is pierceable by $n-1$ points which together with the point a show that $\mathscr{F}$ is n-pierceable.

This completes the proof of part (ii) of the theorem.

3. Proof of part (iii) of the theorem

First we shall show that $h(d, 2) \leqq 3d$ for all d.

Let x_i denote the i-th coordinate of a point x in the coordinate system of $\mathbf{R}^d$ the axes of which are parallel to the edges of the boxes considered. Let $\mathscr{F} \in \Pi^2_{3d} \cap \Delta^d$. If there exists a hyperplane defined by $x_i = constant$ which meets all the members of $\mathscr{F}$, the assertion follows by induction on the dimension. Hence we may assume that for each $i \in \{1, \ldots, d\}$, the projection τ_i of the members of $\mathscr{F}$ into the x_i-axis yields at least one pair of disjoint segments. Without loss of generality we may assume that 0 is the lower bound of the right endpoints of the members of $\tau_i(\mathscr{F})$, while $+1$ is the upper bound of their left endpoints. We denote by $\mathscr{F}_0$ the family consisting of the $2d$ facets (i.e. faces of dimension $d-1$) of the d-cube

$$C^d := \{(x_1, \ldots, x_d) \mid 0 \leqq x_i \leqq 1 \quad \text{for } i = 1, \ldots, d\},$$

and we define $\mathscr{F}^* := \mathscr{F}_0 \cup \mathscr{F}$. From the construction of $\mathscr{F}_0$ it is obvious that

$$\mathscr{F} \in \Pi^2_{3d} \cap \Delta^d \quad \text{implies} \quad \mathscr{F}^* \in \Pi^2_{3d} \cap \Delta^d. \tag{3.1}$$

On the other hand, it is equally obvious that the only pairs of points which pierce all members of $\mathscr{F}_0$ are pairs of opposite vertices of the cube C^d.

For each family $\mathscr{G}$ satisfying $\mathscr{F}_0 \subset \mathscr{G} \subset \mathscr{F}^*$ we define $f(\mathscr{G})$ as the number of pairs of opposite vertices of C^d each of which pierces $\mathscr{G}$. Thus $f(\mathscr{F}_0) = 2^{d-1}$, and our aim is to show that $f(\mathscr{F}^*) > 0$. Let $B_1, B_2, \ldots, B_m$ be boxes in $\mathscr{F}$ such that

$$f(\mathscr{F}_0) > f(\mathscr{F}_0 \cup \{B_1\}) > (\mathscr{F}_0 \cup \{B_1, B_2\}) > \ldots > f(\mathscr{F}_0 \cup \{B_1, B_2, \ldots, B_m\}). \tag{3.2}$$

Clearly $m \leqq 2^{d-1}$, and we assume that the choice of the B_i's is such as to make m maximal. Then naturally

$$f(\mathscr{F}_0 \cup \{B_1, B_2, \ldots, B_m\}) = f(\mathscr{F}^*). \tag{3.3}$$

Now, a vertex $e := (\varepsilon_1, \ldots, \varepsilon_d)$ of C^d belongs to a box B_μ if and only if its coordinates satisfy some system of equations of the type

$$\begin{cases} \varepsilon_{i_1} = \varepsilon_{i_2} = \ldots = \varepsilon_{i_p} \\ \varepsilon_{i_{p+1}} = \varepsilon_{i_{p+2}} = \ldots = \varepsilon_{i_q} = 1 - \varepsilon_{i_1}. \end{cases} \tag{3.4}$$

If we denote by e^* the vertex $e^* := (1, 1, \ldots, 1)$ of C^d, then $e^* - e$ is the vertex of C^d opposite to the vertex e. Hence the pair $\{e, e^* - e\}$ of opposite vertices of C^d pierces the box B if and only if the equations (3.4) are satisfied. The same pair will pierce the boxes $B_1, B_2, \ldots, B_j$ for $i \leqq j \leqq m$ if and only if j systems of the type (3.4) are satisfied or, equivalently, if one — correspondingly larger — system of this type is satisfied.

The vertices of C^d may be interpreted as the d-dimensional affine space over GF(2); hence the system (3.4) determines a flat L in that space. The dimension of L is possibly -1, but certainly not zero since the system (3.4) is invariant under the substitution of $e^* - e$ for e. Therefore the number of solutions of each system of type (3.4) is either 0 or 2^r for some integer $r \geqq 1$. In other words, for each j $(1 \leqq j \leqq m)$ $f(\mathscr{F}_0 \cup \{B_1, \ldots, B_j\})$ is either 0 or a power 2^{r-1} of 2. It follows now from (3.2) and $f(\mathscr{F}_0) = 2^{d-1}$ that $m \leqq d$, hence the family $\mathscr{F}_0 \cup \{B_1, \ldots, B_m\}$ contains at most

$2d+d=3d$ members. But then (3.1) and (3.3) imply that $f(\mathscr{F}^*)>0$ and the proof of $h(d,2)\leqq 3d$ is completed.

Next we shall show that $h(d,2)\geqq 3d$ for odd d, while $h(d,2)\geqq 3d-1$ for even d. Let first $d\geqq 3$ be odd. Changing slightly the notation used above, let from now on C^d denote the d-cube $\{(x_1,\dots,x_d)|-1\leqq x_i\leqq 1$ for $i=1,\dots,d\}$, and let $\mathscr{F}_0$ be the $(2d)$-membered family consisting of the facets of C^d. Let $\mathscr{F}:=\mathscr{F}_0\cup\{B_1,\dots,B_d\}$, where

$$B_i:=\{(x_1,\dots,x_d)\in C^d\,|-1\leqq x_i\leqq 0,\ 0\leqq x_{i+1}\leqq 1\}\quad\text{for}\quad i=1,\dots,d$$

$$(\text{we use}\quad x_{d+1}=x_1).$$

Now $\mathscr{F}$ is not pierceable by two points. Indeed, since $\mathscr{F}_0\subset\mathscr{F}$, the only possible candidates would form an opposite pair of vertices of C^d. Without loss of generality we assume such a pair to be $e=:(1,\varepsilon_2,\dots,\varepsilon_d)$ and $-e$, where $\varepsilon_2,\dots,\varepsilon_d\in\{-1,1\}$. The box B_1 is not pierced by e, hence it has to be pierced by $-e$ and thus $\varepsilon_2=-1$. Considering B_2 we similarly find $\varepsilon_3=1$, etc., hence finally $\varepsilon_d=(-1)^{d-1}=1$, — but then B_d would not be pierced by the pair $\{e,-e\}$. This argument shows also that the points $e_1:=(1,-1,1,-1,\dots,1)$ and $-e_1$ pierce all members of $\mathscr{F}$ except B_d. Similarly $\mathscr{F}\setminus\{B_i\}\in\Pi^2$ for $i=1,2,\dots,d-1$, and we have only to show that $\mathscr{F}\setminus\{B_0\}$ is 2-pierceable if B_0 is one of the facets of C^d — for example the one determined by $x_1=-1$. But this is accomplished by the points e_1 and $e_2:=(0,1,-1,1,\dots,-1)$.

Let now d be even. Using the same notation as for odd d, we consider the family obtained from $\mathscr{F}$ by deleting B_d and by replacing B_0 with the half of B_0 determined by $-1\leqq x_d\leqq 0$. The proof is completely analogous to the one just given, and we omit it.

Both examples become clearer on noting that for each box there are precisely two other boxes in the family that are disjoint from it, and that the whole family forms a circuit (of odd length) in that respect. This aspect of the construction will become even more explicit in the next (and last) step of the proof of (iii).

We shall now show that $h(d,2)$ is odd for all d. Indeed, by the definition of $h(d,2)$ there exists a family $\mathscr{F}$ in Δ^d belonging to $\Pi^2_{h(d,2)-1}$ but not to $\Pi^2_{h(d,2)}$. In other words, there exists a subfamily $\mathscr{G}$ of $\mathscr{F}$ such that

(a) $\mathscr{G}$ contains $h(d,2)$ members;
(b) $\mathscr{G}\notin\Pi^2$;
(c) each proper subfamily of $\mathscr{G}$ belongs to Π^2.

We consider a graph G, the nodes of which are the members of $\mathscr{G}$, while the edges of G are the pairs of disjoint boxes in $\mathscr{G}$. By (a) the graph G has $h(d,2)$ nodes, while (b) means that G is not 2-colorable. (The last assertion follows from the observation that nodes assigned the same color are mutually intersecting boxes which by the proof of part (i) of the theorem, have a common point.) Property (c) means that each proper subgraph of G is 2-colorable. But it is well known (see, for example [6], p. 77) that each minimal not 2-colorable graph is a circuit of odd length. Hence $h(d,2)$ is odd, and the proof of our assertion and of part (iii) of the theorem is completed.

4. Proof of $h(2, 3)=16$

We shall first show that $h(2, 3)\leq 16$. To that effect, let $\mathscr{F}\in\Pi_{16}^3\cap\Delta^2$. As in Section 3 we may assume that the four sides A_0, B_0, C_0, D_0 of the square C^2 are members of $\mathscr{F}$, and that each member of $\mathscr{F}$ meets C^2. In view of the compactness of rectangles, there is no loss of generality in assuming $\mathscr{F}$ finite. Let $\mathscr{F}_0:=\{A_0, B_0, C_0, D_0\}$; using the notation indicated in Figure 1 we have:

(4.1) If $\mathscr{F}_0\subset\mathscr{G}\in\Pi^3$ then $\mathscr{G}$ is 3-pierceable
either (a) by a, c (or b, d) and a third point x of C^2;
or (b) by a, a point y of C_0 different from c, and a point z of D_0 different from c (or by b, or c, or d, and one point in each of the two opposite sides of C^2).

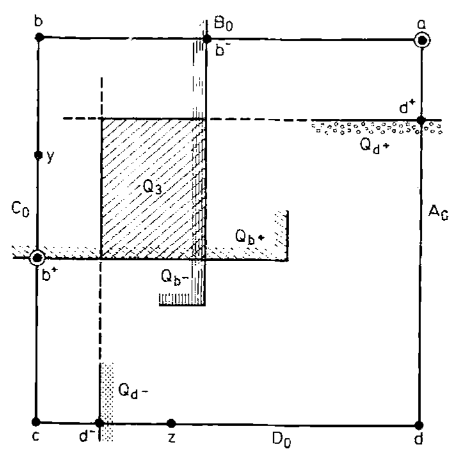

Fig. 1

Let $\mathscr{F}(b^+)$ denote the subfamily of $\mathscr{F}$ consisting of all boxes B satisfying the conditions

(4.2) $$a\notin B \quad \text{and} \quad B\cap C_0\neq\emptyset.$$

$\mathscr{F}(b^+)$ is non-empty since it obviously contains C_0 and D_0. Each element of $\mathscr{F}(b^+)$ meets C_0 in a segment; among the lower endpoints of those segments let b^+ be the highest. Let b^- be the point of B_0 nearest to b for which there exists a box B in $\mathscr{F}$ such that $c\notin B$ and $B\cap\text{conv}\{a, b^-\}=\{b^-\}$. Let $c^+, c^-, d^+, d^-, a^+, a^-$ be defined analogously. We shall first prove the following two assertions:

(4.3a) If $\mathscr{F}$ is not 3-pierceable by a, c and a third point (compare (4.1a)), then there exist members R_1 and R_2 of $\mathscr{F}$ such that already the family $\mathscr{F}_{ac}:=\mathscr{F}_0\cup\{R_1, R_2\}$ is not 3-pierceable by a, c, and a third point.

(4.3b) If $\mathscr{F}$ is not 3-pierceable by a and two points y and z as described in (4.1b), then there exist three members Q_{b^+}, Q_c, Q_{d^-} of $\mathscr{F}$ such that
(1) $a\in Q_{b^+}\cup Q_c\cup Q_{d^-}$;
(2) b^+ is on the lower edge of Q_{b^+}, and d^- is on the left edge of Q_{d^-};

(3) the family $\mathscr{F}_a := \mathscr{F}_0 \cup \{Q_{b^+}, Q_c, Q_{d^-}\}$ is not 3-pierceable by three points $a, y, z,$ as above.

(It should be noted that (4.3) easily implies $h(2,3) \leqq 4+2\cdot 2+4\cdot 3=20$).

As in Section 2, a proof of (4.3a) results at once on considering all the members of $\mathscr{F}$ which contain neither a nor c. In order to prove (4.3b), we choose for Q_{b^+} any rectangle in $\mathscr{F}$ satisfying conditions (1) and (2), and similarly for Q_{d^-}. We also define a point y_0, chosen as b^+ if $b^+ \neq c$, and as $(1-\varepsilon)b^+ + \varepsilon b$ if $b^+ = c$, where $\varepsilon > 0$ is so small that y_0 belongs to all the rectangles B in $\mathscr{F}$ which satisfy

$$(4.2^*) \qquad a \notin B \quad \text{and} \quad B \cap C_0 \neq \emptyset, \{c\}.$$

Similarly we define a point z_0 (as a "neighbor" of d^-).

We assumed that $\mathscr{F}$ is not 3-pierceable by a, y_0, z_0; hence there exists Q_c in $\mathscr{F}$ such that

$$(4.4) \qquad a, y_0, z_0 \notin Q_c.$$

We have only to show that condition (3) is satisfied with this choice of the Q's. Assuming the opposite, (4.4) implies that Q_c contains y or z; without loss of generality we may assume that

$$(4.5) \qquad y \in Q_c.$$

Together with (4.4) this implies that Q_c satisfies (4.2*), hence the case $b^+ = c$ is eliminated. But $b^+ \neq c$ implies (by the definitions of y_c, b^+, and by (4.4)) that $Q_{b^+} \cap Q_c = \emptyset$ and thus, by (4.5), $\{a, y, z\} \cap Q_{b^+} = \emptyset$, — which contradicts the assumption. Therefore (4.3b) holds.

In order to prove now that $\mathscr{F} \in \Pi^3$, we have to distinguish three cases.

Case 1. $\mathscr{F}$ contains a member Q_1 which is disjoint from $A_0 \cup B_0 \cup C_0 \cup D_0$. If we assume that $\mathscr{F}$ is not 3-pierceable by a, c, and a third point, and is equally not 3-pierceable by b, d, and a third point, then — using the notation of (4.3a) — we consider the family

$$\mathscr{G}_1 := \mathscr{F}_{ac} \cup \mathscr{F}_{bd} \cup \{Q_1\} \subset \mathscr{F}.$$

$\mathscr{G}_1$ contains at most 9 members, hence by assumption $\mathscr{G}_1 \in \Pi^3$. But (4.3a) precludes the possibility of 3-piercing $\mathscr{G}_1$ by a triple of type (4.1a), while a triple of type (4.1b) is impossible since $Q_1 \in \mathscr{G}_1$. The contradiction reached completes the proof in case 1.

Case 2. $\mathscr{F}$ contains a member Q_2 which meets exactly one of the sets A_0, B_0, C_0, D_0. The proof in this case is completely analogous to the above, using the families $\mathscr{F}_a$ and $\mathscr{F}_b$ according to (4.3b) and considering the family

$$\mathscr{G}_2 := \mathscr{F}_a \cup \mathscr{F}_b \cup \mathscr{F}_{ac} \cup \mathscr{F}_{bd} \cup \{Q_2\},$$

which contains at most 15 members.

Case 3. Each member of $\mathscr{F}$ meets at least two of the sets A_0, B_0, C_0, D_0. Assuming that $\mathscr{F}$ is not 3-pierceable by a triple of points of type (4.1b), we consider the subfamily

$$\mathscr{G}_3 := \mathscr{F}_a \cup \mathscr{F}_b \cup \mathscr{F}_c \cup \mathscr{F}_d$$

of $\mathscr{F}$. Since $\mathscr{G}_3$ has at most 16 members, $\mathscr{G}_3 \in \Pi^3$. But by its construction $\mathscr{G}_3$ is not 3-pierceable by a triple of type (4.3b); hence we loose no generality in assuming

(4.6) $\mathscr{G}_3$ is 3-pierceable by a, c, and a point x in C^2.

Let Q_3 be the set defined by $\{(x_1, x_2) | d^- \leqq x_1 \leqq b^-,\ b^+ \leqq x_2 \leqq d^+\}$. Then (4.6) implies that $x \in Q_{b^+} \cap Q_{d^+}$, and also $x \in Q_{b^-} \cap Q_{d^-}$; therefore $x \in Q_3$.

Our aim is to show that (4.6) implies also

(4.7) $\mathscr{F}$ is 3-pierceable by a, c, x.

Let $Q \in \mathscr{F}$ be such that $a, c \notin Q$. Then Q meets precisely one of A_0 and B_0, and precisely one of C_0 and D_0. In each case it follows that the lower edge of Q is not above b^+, the upper edge of Q is not below d^+, and similarly for b^- and d^-. Thus $Q \supset Q_3 \ni x$, and (4.7) is established.

This completes the proof of the assertion $h(2, 3) \leqq 16$.

We shall next show that $h(2, 3) \geqq 16$. As before, let a, b, c, d be the vertices of a square C^2 (compare Figure 2), and let A_0, B_0, C_0, D_0 be the sides of C^2. Let

$$\mathscr{F} := \{A_0, A_1, A_2, A_3, B_0, \ldots, B_3, C_0, \ldots, D_0, \ldots, D_3\}$$

be a 16-membered family, the sets A_i being as indicated in Figure 2, and the other sets defined analogously.

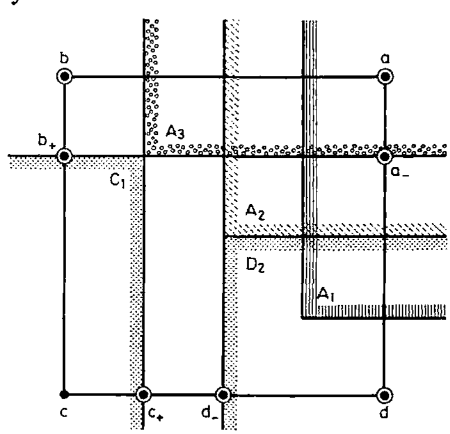

Fig. 2

We note first that $A_1 \cap C_1 = \emptyset$, hence $\mathscr{F}$ is not 3-pierceable by b, d, and a third point. Similarly, a, c, and a third point do not 3-pierce $\mathscr{F}$. Since $\mathscr{F}$ contains the four sides of C^2, it follows that if $\mathscr{F} \in \Pi^3$ it must be 3-pierceable by a vertex of C^2 (for example b) and one point on each of the sides A_0, D_0. But the rectangles A_3, C_1 and D_2 show that $\mathscr{F}$ is not 3-pierceable in this manner; hence $\mathscr{F} \notin \Pi^3$.

On the other hand $\mathscr{F} \in \Pi^3_{15}$. Indeed:

a, $\frac{1}{2}(b+c)$ and d_- 3-pierce $\mathscr{F} \setminus \{B_1\}$, and similarly 3-pierceable are $\mathscr{F} \setminus \{B_3\}$, $\mathscr{F} \setminus \{A_1\}$, $\mathscr{F} \setminus \{A_3\}$, etc.

d, a_+ and c 3-pierce $\mathscr{F}\setminus\{B_2\}$, and similarly 3-pierceable are $\mathscr{F}\setminus\{A_2\}$, etc. b_+, $\frac{1}{2}(c+d)$ and a_- 3-pierce $\mathscr{F}\setminus\{B_0\}$, and similarly 3-pierceable are $\mathscr{F}\setminus\{A_0\}$, etc.

This completes the proof of $h(2, 3)\geqq 16$, and with it the proof of part (iv) of the theorem.

5. Proof of part (V) of the theorem

We shall now prove that $h(d, n)=\aleph_0$ for $d\geqq 2$ and $n\geqq 4$. It is clearly enough to prove this for $d=2$. We choose a positive integer m and denote $k:=k(m):=mn+1$. Let $x_1, \dots, x_k$ be points on the boundary of C^2, dividing the perimeter of C^2 into k equal parts (compare Figure 3). For $j=1, \dots, k$, let A_j be the smallest rectangle (with sides parallel to those of C^2) containing the m points $x_j, x_{j+1}, \dots, x_{j+m-1}$ (where the subscripts are taken modulo k). Let $\mathscr{F}:=\{A_j \mid j=1, \dots, k\}$

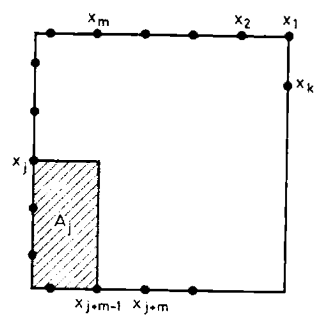

Fig. 3

We note that

$$\frac{m-1}{k}=\frac{m-1}{mn+1}<\frac{m}{mn}\leqq 1/4, \tag{5.1}$$

hence $A_i\cap A_j\neq\emptyset$ for $A_i, A_j\in\mathscr{F}$ if and only if

$$\{x_i, x_{i+1}, \dots, x_{i+m-1}\}\cap\{x_j, x_{j+1}, \dots, x_{j+m-1}\}\neq\emptyset.$$

Therefore no point belongs to more than m members of F and thus $F\notin \Pi^n$.

On the other hand, the family $\mathscr{F}\setminus\{A_j\}$ is n-pierced by $x_{j+m}, x_{j+2m}, \dots, x_{j+nm}$ for each j, hence $\mathscr{F}\in\Pi^n_{k-1}$. Since $\lim_{m\to\infty} k(m)=\infty$, the assertion $h(2, n)=\aleph_0$ follows.

(It may be remarked that the example in Section 4 is a special case of the above construction, with $n=3$, $m=5$ and $k=16$; instead of (5.1), however, we have $\frac{m-1}{k}=1/4$, and correspondingly $A_3\cap C_1\neq\emptyset$, etc.)

In order to complete the proof of the theorem we still have to show that $h(d, n)=\aleph_0$ for $d\geqq 3$, $n\geqq 3$. It is again enough to consider the case $d=3$. Examples are constructed in the same manner as above, taking instead of the perimeter of C^2

the hexagonal line formed by edges of C^3 and having vertices $(1, -1, -1), (1, 1, -1), (-1, 1, -1), (-1, 1, 1), (-1, -1, 1), (1, -1, 1)$.

Instead of the inequality (5.1) we have now

$$\frac{n-1}{k} \leqq 1/3,$$

but this has in the 3-space the same consequences as (5.1) in the plane.

This completes the proof of the theorem.

6. Remarks

Let $\hat{\Delta}^d$ denote the class of all families of translates of the unit cube in $\mathbf{R}^d$. Then clearly $h(\hat{\Delta}^d, n) \leqq h(\Delta^d, n)$. However, an inspection of all the examples in the above proof shows that each could easily be modified to consist of translates of one d-cube. Hence we have

(6.1.) *For all positive integers d and n, $h(\hat{\Delta}^d, n) = h(d, n)$.*

This result naturally leads to the question whether there exists an analogue of (6.1) if one considers families of translates of a convex set different from a cube. More precisely, let K be a convex body in $\mathbf{R}^d$ and let $T^d(K)$ denote the class of all families consisting of translates of K.

The following result is well known (see [1], Section 7, for references to the original papers and for related results);

(6.2) $h(T^d(K), 1) = 2$ *if and only if K is a d-cube.*

We conjecture

(6.3) $h(T^d(K), 2) < \aleph_0$ *if and only K is a convex polytope.*

In the special case $d=2$ we have the following partial result:

(6.4) *If K is a centrally symmetric, strictly convex body in $\mathbf{R}^2$ then $h(T^2(K), 2) = \aleph_0$.*

Indeed, for each such K and for each positive integer $k \geqq 2$, there exists a centrally symmetric convex $(2k)$-gon P circumscribed about K such that the midpoint of each edge of P belongs to K (see [2]). Using the directions of the edges of P it is easy to modify the construction of the "rosette" of [4], p. 17, or [5], p. 11, (compare [3]) to obtain an example showing $h(T^2(K), 2) \geqq k$, which establishes our assertion.

Conjecture (6.3) may probably be extended to the case in which all sets homothetic to a fixed K are allowed instead of just the translates.

As a common generalization of the theorem of Section 1 and of conjecture (6.3), in the special case $n=d=2$, we conjecture:

(6.5) *If D is a set of m directions in the plane, and if Δ_m denotes the class of all families of convex polygons in the plane all edges of which have directions belonging to D, then*

$$h(\Delta_m, 2) \leqq 3m.$$

References

[1] L. DANZER, B. GRÜNBAUM and V. KLEE, Helly's theorem and its relatives, *Proc. Symposia in Pure Math., Vol.* **VII** (Convexity) (1963), 101—180.
[2] M. M. DAY, Polygons circumscribed about closed convex curves, *Trans. Amer. Math. Soc.*, **62** (1947), 315—319.
[3] B. GRÜNBAUM, Common secants for families of polyhedra, *Arch. Math.*, **15** (1964), 76—80.
[4] H. HADWIGER and H. DEBRUNNER, *Kombinatorische Geometrie in der Ebene*, Monographies de l'Enseignement Mathematique, No. **2**, Geneva, 1960.
[5] H. HADWIGER, H. DEBRUNNER and V. KLEE, *Combinatorial geometry in the plane*, New York, 1964.
[6] O. ORE, *The four-color problem*, New York 1967.

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